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Dynamics of localized solutions to the Raman-extended derivative nonlinear Schrödinger equation

Luc Bergé†, Jens Juul Rasmussen and John Wyller‡

Optics and Fluid Dynamics Department, Risø National Laboratory, PO Box 49, DK - 4000 Roskilde, Denmark

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Abstract. Using virial estimates we study the temporal evolution of localized solutions to the Raman-extended derivative nonlinear Schrödinger (R-EDNLS) equation, with a particular emphasis on collapse and non-collapse. By means of a Lagrangian approach, we investigate the possibility of realizing a finite-time self-similar collapse as it appears in solutions to the usual critical nonlinear Schrödinger (CNLS) equation.

1. Introduction

The modulation Ψ of a one-dimensional weakly nonlinear wave train is, under the slowly-varying amplitude approximation, governed by the standard nonlinear Schrödinger (NLS) equation

$$\mathrm{i}\partial_\tau \Psi + q|\Psi|^2\Psi + p\partial_\xi^2\Psi = 0$$

where ξ and τ denote the space and the time coordinates, respectively, while q and p are real constants.

The wave envelope perturbation scheme leading to the NLS equation presupposes a balance between dispersive and nonlinear effects. The outcome of an exact balance is the existence of soliton solutions to the NLS equation. However, when the nonlinear term $q|\Psi|^2\Psi$ is small compared to the dispersion term $p\partial_\xi^2\Psi$, this balance ceases to exist and hence the NLS equation is not appropriate anymore. In terms of the perturbation scheme, we have $q = q(k_0) \sim k_0 - k_c \sim \varepsilon \ll 1 \sim p$ in this case, i.e. $pq \sim \varepsilon$, where ε is a small expansion parameter, k_0 is the carrier wave number and k_c the critical wave number satisfying $q(k_0 = k_c) = 0$. Since the plane waves of the NLS equation are modulationally stable (unstable) if $pq < 0$ ($pq > 0$), the regime $pq \sim \varepsilon \ll 1$, for which the nonlinearity acts against the dispersion to a small extent only, received the name of *marginal stable* state [1]. In order to study the influence of the nonlinearity on the wave propagation in this regime, a more detailed perturbation expansion has to be employed. In fact, as shown in [2], one has to assume that the perturbed field quantities are of order $\varepsilon^{1/2}$ (instead of ε as in the NLS case); this yields the need to take some additional contributions into account and one obtains the generalized NLS equation

$$\mathrm{i}\partial_\tau \Psi + \partial_\xi^2\Psi = q_1|\Psi|^2\Psi + q_2|\Psi|^4\Psi + \mathrm{i}q_3\Psi\partial_\xi(|\Psi|^2) + \mathrm{i}q_4|\Psi|^2\partial_\xi\Psi \qquad (1.1)$$

† Permanent address: Commissariat à l'Energie Atomique, Centre d'Etudes de Limeil-Valenton, 94195 Villeneuve-St-Georges Cedex, France.

‡ Permanent address: Department of Mathematics, Narvik Institute of Technology, PO Box 385, N-8501 Narvik, Norway.

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($q_1, \ldots, q_4$ are real constants) as a model for nonlinearly modulated wave trains in the marginal stable regime. This phenomenon can occur in different physical systems such as Stokes waves in fluids of finite depth [3], ion acoustic waves in a two-electron temperature plasma [4] and ion acoustic waves in a plasma composed of electrons and of positive and negative ions [5].

In nonlinear optics the extended version of the generalized NLS equation (1.1)

$$\mathrm{i}\partial_\tau \Psi + \partial_\xi^2 \Psi = q_1|\Psi|^2\Psi + q_2|\Psi|^4\Psi + (q_5 + \mathrm{i}q_3)\Psi\partial_\xi(|\Psi|^2) + \mathrm{i}q_4|\Psi|^2\partial_\xi\Psi \tag{1.2}$$

where $q_1, \ldots, q_5$ are real constants, can be derived in a systematic way by means of the reductive perturbation scheme as a model for single mode propagation [6]. In the context of waveguides as optical fibres, τ usually corresponds to the propagation distance of the electric field envelope Ψ of an optical beam along the fibre, ξ plays the role of the time, the terms $q_1|\Psi|^2\Psi$ and $q_2|\Psi|^4\Psi$ model the nonlinear 'Kerr' effect, $\mathrm{i}q_3\Psi\partial_\xi(|\Psi|^2)$ and $\mathrm{i}q_4|\Psi|^2\partial_\xi\Psi$ the nonlinear dispersion contributions and the new term $q_5\Psi\partial_\xi(|\Psi|^2)$ a nonlinear optical delay effect called the Raman response. In typical situations, the quintic nonlinearity of (1.2) is a saturating nonlinearity satisfying $q_1q_2 < 0$ with, typically, $|q_2| \ll |q_1|$, and we henceforth assume $q_2 < 0$. Moreover, according to Kodama and Hasegawa [6] the Raman coefficient q_5 is positive. From now on we refer to equation (1.2) as the higher-order nonlinear Schrödinger (HNLS) equation.

The HNLS equation can be simplified by using the following set of transformations [7]. First, by introducing the point transformation

$$\begin{aligned}&\Psi(\xi, \tau) = v(x_1, t_1)\exp[\mathrm{i}(\chi x_1 - \chi^2 t_1)]\\ &x_1 = \xi - 2\chi\tau \qquad t_1 = -\tau\end{aligned} \tag{1.3}$$

with $q_1 = \chi q_4$, $q_4 \neq 0$, we obtain the evolution equation

$$\mathrm{i}\partial_{t_1} v = \partial_{x_1}^2 v + c|v|^4 v + \mathrm{i}a|v|^2\partial_{x_1} v + \mathrm{i}bv^2\partial_{x_1} v^* + \mathrm{d}v\partial_{x_1}(|v|^2) \tag{1.4}$$

where $a = -(q_3 + q_4)$, $b = -q_3$, $c = -q_2$ and $d = -q_5$. Then, by employing the $U(1)$-gauge transformation

$$\begin{aligned}&w = v\exp\left[\mathrm{i}\tfrac{1}{2}b\psi\right] \qquad \partial_{x_1}\psi = |w|^2\\ &\partial_{t_1}\psi = \mathrm{i}(w\partial_{x_1}w^* - w^*\partial_{x_1}w) + \tfrac{1}{2}b|w|^4\end{aligned} \tag{1.5}$$

one gets

$$\mathrm{i}\partial_{t_1} w = \partial_{x_1}^2 w + \mathrm{i}\tilde{b}|w|^2\partial_{x_1} w + \mathrm{d}w\partial_{x_1}(|w|^2) + \tilde{c}|w|^4 w \tag{1.6}$$

with $\tilde{b} = a - b$ and $\tilde{c} = c - \frac{1}{4}b(2b - a)$. Note that the 'stream function' ψ in (1.5) is well defined, even in the case

$$d \neq 0 \qquad \tilde{c} \neq 0$$

since w satisfies the conservation law

$$\partial_{t_1}(|w|^2) = \partial_{x_1}\left[\mathrm{i}(w\partial_{x_1}w^* - w^*\partial_{x_1}w) + \tfrac{1}{2}b|w|^4\right].$$

Finally, equation (1.6) can be rewritten in the standard form

$$\partial_t u + \mathrm{i}\partial_x^2 u + |u|^2\partial_x u + \mathrm{i}\gamma u\partial_x(|u|^2) + \mathrm{i}\sigma|u|^4 u = 0 \tag{1.7}$$

by means of the scaling transformation

$$u = w \qquad x = -\tilde{b}x_1 \qquad t = \tilde{b}^2 t_1. \tag{1.8}$$

Returning to the original coefficients of the HNLS equation (1.2), we can identify the real parameters σ and γ of (1.7), as defined by

$$\sigma = \frac{q_3(q_3 - q_4) - 4q_2}{4q_4^2} \qquad \gamma = -\frac{q_5}{q_4}. \tag{1.9}$$

From a practical viewpoint, the coefficient q_3 should be identically equal to q_4, leading to $\sigma = -q_2/q_4^2$. As the quintic nonlinearity is classically of second order as compared with the cubic one, we should thus in principle consider the limit $\sigma \ll 1(|q_2| \to 0)$. However, even in the previous limit, it is natural to retain this quintic contribution issued from the Kerr nonlinearity, since the dominant cubic nonlinearity has formally been absorbed by applying the point transformation (1.3). On the other hand, as will be seen further on, this quintic nonlinearity plays a major role in the fate of optical pulses, in the sense that it may be responsible for the collapse (i.e. the total self-focusing) of the pulses, provided that the latter initially exhibit a sufficiently large amplitude. Note furthermore that assuming $q_4 > 0$ leads to $\gamma < 0$.

We refer to equation (1.7) as the Raman-extended derivative nonlinear Schrödinger (R-EDNLS) equation. The mathematical properties of this equation are well known for special cases. When $\sigma = \gamma = 0$, it is completely integrable by inverse scattering transform [8, 9]. For only $\gamma = 0$ ($\sigma \neq 0$) it forms a Hamiltonian system with three conservation laws (conservation of mass, linear momentum and energy) [1], while in the general case the Hamiltonian structure has been lost, but still the mass integral $N \equiv \int |u|^2 \, \mathrm{d}x$ is conserved.

In spite of this 'loss' of invariants, we are able to analyse qualitatively the dynamics of localized solutions in the non-integrable cases using virial analytical techniques and boundedness of the Hamiltonian, in a way analogous to the critical nonlinear Schrödinger equation (CNLS)

$$\partial_t u + \mathrm{i}\partial_x^2 u + \mathrm{i}\sigma |u|^4 u = 0.$$

This is the objective of the present paper. In particular, we investigate the influence of the derivative nonlinear term, $|u|^2\partial_x u$, and/or the Raman term, $\mathrm{i}\gamma u \partial_x(|u|^2)$, on the collapse dynamics described by the CNLS equation when $\sigma > 0$. The result has general interest in connection with the question of arresting the collapse in nonlinear Schrödinger models by means of higher-order contributions. The R-EDNLS equation may be considered as a model in nonlinear optics as discussed above and our results apply to the development and self-contraction of optical pulses propagating along a waveguide. More generally the equation without the Raman term ($\gamma = 0$) governs the evolution of wave envelopes near the marginal stable state and our results apply to the dynamics of localized wavepackets under these conditions as well.

Even though virial-type relations can be derived in the R-EDNLS case, the complexity of both the Hamiltonian and the virial identity prevents us from getting precise estimates and hence exact results in contrast to the CNLS case. Nevertheless, in the context of the R-EDNLS equation, it is possible to predict the occurrence of collapsing solutions for $\sigma > 0$ by using test functions analogous to the ones employed in the framework of the CNLS equation, these test functions being inserted into both the Lagrangian (when the latter exists) and the virial identity of equation (1.7). The results of our analysis can then be summarized as follows. In section 2, we derive the equations governing the motion of the centre of mass and the virial identity. In section 3 we show that no localized solution exists in the presence of the Raman effect ($\gamma \neq 0$). In the opposite case $\gamma = 0$, we prove in section 4 that equation (1.7) admits localized solutions which remain bounded and are expected to exist globally in time for a positive Hamiltonian, provided that the mass integral lies below a threshold value. When these conditions are not fulfilled, we use the virial identity of the R-EDNLS equation together with the assumption that the leading order part of the solution can be represented as a generalized self-similar waveform to show that a finite-time collapse may occur when $\gamma = 0$, while the presence of the Raman effect can possibly arrest the collapse. The limitations of this procedure are discussed in section 5.

2. The centre of mass and the virial identity of time-dependent solutions to the R-EDNLS equation

The EDNLS equation defined by equation (1.7) without the Raman response,

$$\partial_t u + \mathrm{i}\partial_x^2 u + |u|^2\partial_x u + \mathrm{i}\sigma|u|^4 u = 0 \tag{2.1}$$

possesses three conservation laws [1], namely

$$\partial_t U_i + \partial_x P_i = 0 \qquad i = 1, 2, 3 \tag{2.2}$$

where the densities U_i and the fluxes P_i take the form

$$U_1 = |u|^2 \tag{2.3}$$
$$P_1 = \mathrm{i}(u^*\partial_x u - u\partial_x u^*) + \tfrac{1}{2}|u|^4$$
$$U_2 = \tfrac{1}{2}\mathrm{i}(u^*\partial_x u - u\partial_x u^*) \tag{2.4}$$
$$P_2 = \tfrac{1}{2}\mathrm{i}(u\partial_t u^* - u^*\partial_t u) + \tfrac{1}{2}|u|^4$$
$$U_3 = |\partial_x u|^2 - \tfrac{1}{4}\mathrm{i}|u|^2(u\partial_x u^* - u^*\partial_x u) - (\sigma/3)|u|^6 \tag{2.5}$$
$$P_3 = -(\partial_t u\partial_x u^* + \partial_t u^*\partial_x u) + \mathrm{i}\tfrac{1}{4}|u|^2(u\partial_t u^* - u^*\partial_t u).$$

Here $i = 1$ corresponds to the conservation of mass, $i = 2$ leads to the conservation of the linear momentum and $i = 3$ to the conservation of energy. We can then express P_1 in terms of U_1 and U_2:

$$P_1 = 2U_2 + \tfrac{1}{2}U_1^2. \tag{2.6}$$

Moreover, the EDNLS equation (2.1) forms a Hamiltonian system, where the Hamiltonian H is given by

$$H \equiv \int U_3\,\mathrm{d}x = \|\partial_x u\|_2^2 - (\sigma/3)\|u\|_6^6 - \tfrac{1}{4}\mathrm{i}\int |u|^2(u\partial_x u^* - u^*\partial_x u)\,\mathrm{d}x \tag{2.7}$$

with

$$\|u\|_6^6 \equiv \int |u|^6\,\mathrm{d}x \qquad \|\partial_x u\|_2^2 \equiv \int |\partial_x u|^2\,\mathrm{d}x$$

and it is assumed that $u \in W^{1,2}$ at least locally in time t. Here and in the following, we use the standard notation of the theory of Sobolev spaces, in particular, $\|\cdot\|_p$ represents the usual L^p norm ($\|f\|_p \equiv (\int |f|^p\,\mathrm{d}x)^{1/p}$ for any L^p-integrable function f).

When we include the effect of the Raman term, i.e. when u obeys the R-EDNLS equation (1.7) with $\gamma \neq 0$, only the mass density $U_1 \equiv |u|^2$ leads to the conserved integral $\int U_1\,\mathrm{d}x$, and the evolution of the density U_2 is given by

$$\partial_t U_2 + \partial_x P_2 - \tfrac{1}{2}\gamma\partial_x^2(U_1^2) + \gamma[\partial_x U_1]^2 = 0. \tag{2.8}$$

Let $N \equiv \int |u|^2\,\mathrm{d}x \equiv \|u\|_2^2$ denote the 'total mass' of the wave field. The centre of mass (CM) $\langle x\rangle$ of the wave is defined by

$$\langle x\rangle \equiv N^{-1}\int x|u|^2\,\mathrm{d}x. \tag{2.9}$$

Then, by using the relations (2.2), (2.3), (2.6), (2.7) and (2.8), we easily derive the equations for the 'velocity' and the 'acceleration' of the CM, reading respectively as

$$N\partial_t\langle x\rangle = 2\int U_2\,\mathrm{d}x + \tfrac{1}{2}\int U_1^2\,\mathrm{d}x \tag{2.10}$$

$$N\partial_t^2\langle x\rangle = -2\gamma\int(\partial_x U_1)^2\,\mathrm{d}x - \mathrm{i}\int U_1(u^*\partial_x u - u\partial_x u^*)_x\,\mathrm{d}x. \tag{2.11}$$

Furthermore, the 'mean square radius' $\langle x^2\rangle$ of the wave field is defined by

$$\langle x^2\rangle \equiv N^{-1}\int x^2|u|^2\,\mathrm{d}x = N^{-1}\int x^2 U_1\,\mathrm{d}x$$

in such a way that by differentiating this integral twice in time and by using the relations (2.2), (2.3), (2.6), (2.7) and (2.8), we can establish the following virial identity

$$\begin{aligned} N\partial_t^2\langle x^2\rangle &= 8H + \tfrac{2}{3}\|u\|_6^6 + 2\mathrm{i}\int xU_1\partial_x(u\partial_x u^* - u^*\partial_x u)\,\mathrm{d}x - 4\gamma\int x\{\partial_x U_1\}^2\,\mathrm{d}x \\ &= 8H + \partial_t\left(\int xU_1^2\,\mathrm{d}x\right) - 4\gamma\int x\{\partial_x U_1\}^2\,\mathrm{d}x \end{aligned} \tag{2.12}$$

where H is still given by (2.7) and remains conserved for $\gamma = 0$ only.

3. General properties of stationary and self-similar solutions

Particular solutions of the EDNLS equation (1.7) may be sought under the form of stationary states or of travelling-wave structures propagating along the x-axis with constant velocity. In this respect, such travelling-wave solutions of the EDNLS equation (2.1) have already been investigated in [1] and [7]. Their existence follows from the property that there is a coupling between the constant velocity of the solution and its amplitude, which is due to the nonlinear derivative of the EDNLS equation. More generally, regarding any time-dependent solutions, it can be noted that this derivative term introduces a nonlinear convection of the localized waveforms. Nevertheless, for the sake of simplicity, we will limit our investigation to the simpler class of stationary solutions, i.e. steady-state solutions carrying a zero velocity, bearing in mind that the latter, consisting of localized soliton-type structures, can be viewed as being good candidates for the initial value problem associated with the R-EDNLS equation.

3.1. Stationary solutions

The expression for the stationary wave solution to the R-EDNLS equation is given by

$$u(x,t) = \phi(x)\exp[-\mathrm{i}\lambda t]. \tag{3.1}$$

By inserting expression (3.1) into (1.7), we find that ϕ obeys the ordinary differential equation (ODE)

$$-\mathrm{i}\lambda\phi + \mathrm{i}\phi'' + |\phi|^2\phi' + \mathrm{i}\sigma|\phi|^4\phi + \mathrm{i}\gamma\phi(|\phi|^2)' = 0 \tag{3.2}$$

where $\phi' \equiv \mathrm{d}\phi/\mathrm{d}x$ and $\phi'' \equiv \mathrm{d}^2\phi/\mathrm{d}x^2$.

3.1.1. The case $\gamma = 0$. First, let us assume $\gamma = 0$. Setting $\sigma > 0$, we can then prove the non-existence of non-zero, smooth, localized solutions of (3.2) for $\lambda \leqslant 0$ by arguing by contradiction. Indeed, let us suppose that there exists a $\phi \neq 0 \in W^{1,2}$ satisfying (3.2) for $\lambda \leqslant 0$. By multiplying equation (3.2) by $x(\phi^*)'$, retaining the imaginary part of the result and integrating over space, we have

$$-\lambda\|\phi\|_2^2 + \|\phi'\|_2^2 + \tfrac{1}{3}\sigma\|\phi\|_6^6 = 0 \tag{3.3}$$

where $\|\phi\|_6^6$ is finite by virtue of the Sobolev inequality, from which we arrive at a contradiction. The solutions ϕ may thus be localized for $\lambda > 0$ only. In the following we assume that this requirement is always fulfilled and we refer to equation (3.2) as the ground-state equation. The latter, moreover, obeys the scale invariance

$$x \to \lambda^{1/2}x \qquad \phi \to \lambda^{1/4}\phi$$

so that the mass integral $N_s \equiv \int |\phi|^2 \, \mathrm{d}x \equiv \|\phi\|_2^2$ is independent of λ.

Let us now consider the acceleration equation (2.11) when u is given by (3.1) for $\gamma = 0$. By multiplying equation (3.2) with $|\phi|^2\phi^*$, retaining only the real part and integrating over space, we obtain

$$\mathrm{i}\int |\phi|^2(\phi^*\phi' - \phi(\phi')^*)' \, \mathrm{d}x = 0$$

and hence relation (2.11) simply reduces to

$$\partial_t^2\langle x\rangle = 0$$

as expected.

Next, we consider the virial identity (2.12) in the same situation and proceed as follows. By multiplying equation (3.2) by ϕ^* and integrating the imaginary part of the result over space, we obtain the identity

$$-2\lambda\|\phi\|_2^2 - 2\|\phi'\|_2^2 + \mathrm{i}\int |\phi|^2(\phi(\phi^*)' - \phi^*\phi') \, \mathrm{d}x + 2\sigma\|\phi\|_6^6 = 0. \tag{3.4}$$

Then we multiply (3.2) by $x|\phi|^2\phi^*$ and retain the real part of the space integrated result to find

$$2\mathrm{i}\int x|\phi|^2((\phi^*)'\phi - \phi^*\phi')' \, \mathrm{d}x = -\tfrac{2}{3}\|\phi\|_6^6. \tag{3.5}$$

By inserting (3.1), (3.3) and (3.6) into the energy integral (2.7), we get

$$H = 0 \tag{3.6}$$

and finally, by plugging (3.1), (3.5) and (3.6) into (2.12), we readily obtain

$$\partial_t^2\langle x^2\rangle = 0 \tag{3.7}$$

showing that computed on the ground states defined by the stationary solutions (3.1), both the energy and the virial relation reduce to zero. The result (3.7) is of course expected to be satisfied for a stationary solution. The property $H = 0$, in addition to the above scale invariance, makes the properties of the ground-state solutions to the EDNLS equation (2.1) very similar to the ones of the CNLS equation.

3.1.2. The case $\gamma \neq 0$. Now, let us assume $\gamma \neq 0$. By multiplying (3.2) by $(\phi^*)'$, integrating only the imaginary part, one gets

$$\gamma \int [(|\phi|^2)']^2 \,\mathrm{d}x = 0. \tag{3.8}$$

This result (3.8) implies that the ODE (3.2) cannot possess any *real* localized ground states different from zero, otherwise ϕ should reduce to a non-zero constant, which is impossible. This remark applies in particular to the case when the derivative term $|u|^2\partial_x u$ in (1.7), or equivalently $|\phi|^2\phi'$ in (3.2), is disregarded, leading *a priori* to real stationary solutions.

In this connection, we can extend this latter result to the non-existence of any non-zero localized *complex-valued* stationary solutions even when retaining the derivative contribution $|u|^2\partial_x u$ in (1.7). Indeed, let us suppose the contrary and write ϕ in the form $\phi = \rho \exp[\mathrm{i}\theta]$. Then, equation (3.2) decomposes into two equations:

$$-\lambda\rho + \rho'' - \rho(\theta')^2 + \rho^3\theta' + \sigma\rho^5 + \gamma\rho(\rho^2)' = 0 \tag{3.9}$$

$$2\rho'\theta' + \rho\theta'' = \rho^2\rho'. \tag{3.10}$$

Multiplying (3.10) by ρ, then integrating the resulting equation enables us to determine the first derivative of the phase as follows,

$$\theta' = \frac{1}{4}\rho^2 + \frac{C}{\rho^2} \tag{3.11}$$

where C denotes an integration constant. Substituting this result into equation (3.9) yields

$$-\left(\lambda - \tfrac{1}{2}C\right)\rho + \rho'' + \left(\sigma + \tfrac{3}{16}\right)\rho^5 + \gamma\rho(\rho^2)' - \frac{C^2}{\rho^3} = 0 \tag{3.12}$$

for which it is clear that the constraint $C = 0$ must follow when regarding smooth, localized solutions, i.e. solutions vanishing asymptotically like $\rho(x \to \pm\infty) = 0$ together with all their derivatives. In this case, multiplying equation (3.12) by ρ' and integrating the resulting equation leads to $(\gamma/2)\int[(\rho^2)']^2\,\mathrm{d}x = 0$, which implies that $\rho =$ constant for all x. Consequently, equation (1.7) has no non-trivial localized ground-state solutions when $\gamma \neq 0$.

3.2. Self-similar solutions

We now consider proper self-similar solutions of equation (1.7). The self-similar transformation of the R-EDNLS equation is given by (see, e.g., [7])

$$u(x,t) = t^{-\frac{1}{4}(1+\mathrm{i}\beta)} f(\xi) \qquad \xi = 2xt^{-\frac{1}{2}} \tag{3.13}$$

and, after plugging it into the R-EDNLS equation, it yields the ODE

$$-\tfrac{1}{4}(1+\mathrm{i}\beta)f - \tfrac{1}{2}\xi f' + \mathrm{i}4f'' + 2|f|^2 f' + \mathrm{i}2\gamma f(|f|^2)' + \mathrm{i}\sigma|f|^4 f = 0. \tag{3.14}$$

Here it is assumed that $f \in W^{1,2}$. The acceleration equation for the CM now takes the form

$$N\partial_t^2\langle x\rangle = -4\gamma t^{-3/2}\int[(|f|^2)']^2\,\mathrm{d}\xi - 2\mathrm{i}t^{-3/2}\int |f|^2(f^*f' - f(f^*)')'\,\mathrm{d}\xi \tag{3.15}$$

that can be simplified by means of (3.14) in the following way. We multiply equation (3.14) by $|f|^2 f^*$ and integrate the real part of the result over space to get the identity

$$2\mathrm{i}\int |f|^2(f^*f' - f(f^*)')'\,\mathrm{d}\xi = \tfrac{1}{8}\|f\|_4^4 \tag{3.16}$$

where $\|f\|_4^4 \equiv \int |f|^4 \, d\xi$.

Inserting (3.16) into (3.15), one readily obtains the equation

$$N\partial_t^2\langle x\rangle = -4t^{-3/2}\left(\gamma\int[(|f|^2)']^2\,d\xi + \tfrac{1}{32}\|f\|_4^4\right) \tag{3.17}$$

for the acceleration of the CM of localized self-similar waves. Thus, for $\gamma > 0$, the speed $\partial_t\langle x\rangle$ always decreases with time t, while for $\gamma < 0$ there is a value $\gamma = \gamma_{cr}$ given by

$$\gamma_{cr} \equiv -\frac{1}{32}\frac{\|f\|_4^4}{\int[(|f|^2)']^2\,d\xi} \tag{3.18}$$

such that the acceleration of the CM is identically equal to zero, implying therefore that the self-similar wave propagates with a constant speed.

4. Collapse dynamics

The previous property of a vanishing Hamiltonian for $\gamma = 0$ and the fact that the mass integral is independent of λ for the ground-state solution are quite interesting in the sense that these results are similar to the ones obtained from the critical nonlinear Schrödinger (CNLS) equation

$$\partial_t u + i\partial_x^2 u + i\sigma|u|^4 u = 0 \tag{4.1}$$

which is refound from equation (1.7) by disregarding the Raman term ($\gamma = 0$) and the derivative contribution $|u|^2\partial_x u$. As reviewed in [10], equation (4.1) is a Hamiltonian system possessing the conserved energy integral

$$H_{\mathrm{CNLS}} = \|\partial_x u\|_2^2 - (\sigma/3)\|u\|_6^6 \tag{4.2}$$

that vanishes on the ground-state solution of (4.1) in the form (3.2); this solution is localized for $\lambda > 0$. Another property deduced from the Hamiltonian (4.2) is that the time-dependent solutions to equation (4.1) are bounded for all times provided that the L^2 norm N is smaller than a threshold value N_0 which is just the mass integral N computed on the ground-state solution [10, 11]. In this case H has to be positive. Let us here show that an analogous property characterizes the solutions of the EDNLS equation. First of all, we make use of the simple inequality

$$-|a| \leqslant a \leqslant |a| \qquad \forall a$$

to display that the last contribution in the expression (2.7) can be expressed in terms of the two other ones $\|\partial_x u\|_2^2$ and $\|u\|_6^6$. Indeed, by doing so, we get

$$H \leqslant \|\partial_x u\|_2^2 - (\sigma/3)\|u\|_6^6 + \tfrac{1}{2}\int |u|^2|u||\partial_x u|\,dx$$

such that when employing the Schwarz inequality together with the simple estimate

$$\left(\frac{a}{\varepsilon} - \varepsilon b\right)^2 \geqslant 0 \qquad \forall a, b, \varepsilon \tag{4.3}$$

we find

$$H \leqslant \|\partial_x u\|_2^2 - (\sigma/3)\|u\|_6^6 + \frac{1}{4}\left(\varepsilon^2\|u\|_6^6 + \frac{1}{\varepsilon^2}\|\partial_x u\|_2^2\right). \tag{4.4}$$

Setting $\varepsilon^2 = \frac{4}{3}\sigma$ for $\sigma > 0$, H is simply bounded from above by

$$H \leqslant \left(1 + \frac{3}{16\sigma}\right)\|\partial_x u\|_2^2.$$

Now, we use the well known Sobolev (Gagliardo–Nirenberg) inequality

$$\|u\|_6^6 \leqslant K\|\partial_x u\|_2^2 \|u\|_2^4 \qquad K = \text{constant} > 0 \tag{4.5}$$

to bound H from below: employing again $-|a| \leqslant a$, (4.3), together with (4.5), we rearrange all the terms to obtain

$$\begin{aligned} H &\geqslant \left(1 - \frac{1}{4\varepsilon^2}\right)\|\partial_x u\|_2^2 - \left(\frac{\varepsilon^2}{4} + \frac{\sigma}{3}\right)\|u\|_6^6 \\ &\geqslant \left[1 - \frac{1}{4\varepsilon^2} - \left(\frac{\varepsilon^2}{4} + \frac{\sigma}{3}\right)K\|u\|_2^4\right]\|\partial_x u\|_2^2. \end{aligned} \tag{4.6}$$

Setting $\varepsilon^2 = 1$ in (4.6), we get

$$H \geqslant \left(\frac{3}{4} - \frac{(3+4\sigma)}{12}K\|u\|_2^4\right)\|\partial_x u\|_2^2$$

from which we deduce that $\|\partial_x u\|_2^2$ always remains bounded provided that the mass N lies below a critical value N_c defined by

$$N_c \equiv \frac{3}{\sqrt{(4\sigma+3)K}} \tag{4.7}$$

where, following [11], the best constant K optimizing the Sobolev inequality (4.5) is given by $K_{best} \equiv 3/(\sigma N_0^2)$. In this case, $\|u\|_6^6$ can be controlled by virtue of the constancy of H and therefore remains bounded in turn. Solutions $u(x,t)$ satisfying $N < N_c$ can thus be expected to exist globally in time; in this case H is positive. In the opposite case $H \leqslant 0$, the estimate (4.6) shows that N necessarily satisfies $N \geqslant N_c$ and we cannot control the gradient and the L^6 norm in the Hamiltonian *a priori* any longer. Note that even for $H > 0$, we may have $N > N_c$, so that also in this situation, the previous norms cannot be controlled. In the context of the CNLS (4.1), such a situation gives rise to blowing-up structures, i.e. solutions that collapse in finite time with a constant mass and with a diverging mass density U_1. When collapse occurs, the diverging solutions turn out to exhibit a self-similar shape $|u|^2 = [1/a(t)]f(x/a(t))$ near the collapse singularity. Here $a(t)$ denotes the typical width of these solutions that are expected to blow-up at a finite time $t_c < +\infty$ with $a(t) \to 0$ as $t \to t_c$. For the CNLS equation (4.1), the blow-up dynamics simply results from the vanishing of the mean square radius $\langle x^2\rangle$ as $t \to t_c$. This vanishing easily follows from the standard virial identity

$$N\partial_t^2\langle x^2\rangle = 8H \tag{4.8}$$

where H is given by (4.2), under the sufficient requirement $H < 0$.

However, in the present scope of the R-EDNLS equation, the equality (4.8) is supplemented by two additional contributions, namely $\chi \equiv \partial_t\left(\int x|u|^4\,\mathrm{d}x\right)$ and $\Gamma \equiv -4\gamma\int x\{\partial_x(|u|^2)\}^2\,\mathrm{d}x$, the effects of which cannot be directly estimated in the expression (2.12). Instead of finding exact estimates of this latter relation (2.12), we suppose—by analogy with the CNLS equation—that if collapse occurs, the singular part of the solution has to behave with an exact self-similar shape as $t \to t_c$. This amounts to approaching the dynamics in time of a typical radius $a(t)$ of a localized structure which is *a priori* self-similar (the property according to which equation (1.7) admits self-similar-type solutions can easily be justified in view of section 3.2). In the simple case of the CNLS equation, $a(t)$ may be of the form $\sqrt{t_c - t}$ or linear in time with $a(t) \sim (t_c - t)$, depending on the initial data. In the following, we assume $H < 0$, leading to collapsing solutions in the CNLS case, and investigate the effects of the additional derivative contributions in (2.12) for exactly self-similar solutions. In the simpler CNLS case, these behaviours of $a(t)$ may directly be deduced by integrating (4.8) twice in time and/or by a so-called variational procedure based on the Lagrangian density

$$\mathcal{L}_{\mathrm{CNLS}} = -\tfrac{1}{2}\mathrm{i}(u^*\partial_t u - u\partial_t u^*) - |\partial_x u|^2 + (\sigma/3)|u|^6 \tag{4.9}$$

from which the CNLS equation (4.1) can be derived when expanding $\delta L/\delta u^* = 0$ with $L = \int \mathcal{L}_{\mathrm{CNLS}}\,\mathrm{d}x$. The variational (or Lagrangian) procedure then consists in estimating the time evolution of the width $a(t)$ entering the test function

$$u(x,t) = \frac{1}{\sqrt{a(t)}}\tilde{\Phi}(\xi)\exp\left[-\frac{\mathrm{i}\dot{a}(t)}{4a(t)}x^2\right] \qquad \xi \equiv \frac{x}{a(t)} \tag{4.10}$$

with $\dot{a} \equiv \mathrm{d}a/\mathrm{d}t$. This test function (4.10) is exactly self-similar in the sense that $\tilde{\Phi}$ does not explicitly depend on time and the space dependences in the phase assure the self-consistency of the self-similarity rescaling $x \to x/a(t)$ with the kinetic contributions of L and with the mass continuity equation (2.1) [12]. Plugging (4.10) into equation (4.1), one gets the self-similarly transformed problem:

$$\epsilon\xi^2\tilde{\Phi} + \partial_\xi^2\tilde{\Phi} + \sigma|\tilde{\Phi}|^4\tilde{\Phi} = 0 \tag{4.11}$$

$$\epsilon = -\tfrac{1}{4}a^3\ddot{a}. \tag{4.12}$$

In addition, introducing solution (4.10) into $L = \int \mathcal{L}_{\mathrm{CNLS}}\,\mathrm{d}x$ then yields $L = \overline{L}(a, \dot{a}, \ddot{a})$ with

$$\overline{L} = -\frac{a\ddot{a}}{4}I - \frac{E\{\tilde{\Phi}\}}{a^2} \tag{4.13}$$

where $I \equiv \int \xi^2|\tilde{\Phi}|^2\,\mathrm{d}\xi$ and

$$E\{\tilde{\Phi}\} = \|\partial_\xi\tilde{\Phi}\|_2^2 - (\sigma/3)\|\tilde{\Phi}\|_6^6$$

is nothing else but the energy integral (4.2) of CNLS expressed in terms of the self-similar function $\tilde{\Phi}$. Writing the variational equations

$$\partial_t\left(\frac{\partial\overline{L}}{\partial\dot{a}}\right) - \partial_t^2\left(\frac{\partial\overline{L}}{\partial\ddot{a}}\right) = \frac{\partial\overline{L}}{\partial a} \tag{4.14}$$

provides the dynamical system

$$\epsilon = -\frac{a^3\ddot{a}}{4} = -\frac{E\{\tilde{\Phi}\}}{I} \tag{4.15}$$

which predicts a vanishing of $a(t)$ with $a(t) \sim \sqrt{t_c - t}$ for $E\{\tilde{\Phi}\} < 0$ (this means that computed with $\tilde{\Phi}$, the nonlinear contribution of H_{CNLS} dominates over the dispersion) or with $a(t) \sim (t_c - t)$ for $E\{\tilde{\Phi}\} = 0$. In this latter case, one obtains $\epsilon = 0$ and when we introduce the ground-state substitution

$$u(x,t) \to u(x,t)\exp\left[-\mathrm{i}\lambda\int_0^t \mathrm{d}u/a^2(u)\right] \tag{4.16}$$

into (4.1) with $u(x,t)$ given by (4.10) and $\lambda > 0$, $\tilde{\Phi}$ simply reduces to the standard stationary solution of CNLS carrying a zero energy ($H\{\Phi\} = 0$). As announced above, the dynamical system (4.15) can also be established from the virial identity (4.8). Indeed, let us now insert the substitution (4.10) into relation (4.8): by doing so, we readily get $\partial_t^2(a^2 I) = 8H$ where H defined by (4.2) expands in the form

$$H \to \frac{1}{a^2}[\|\partial_\xi\tilde{\Phi}\|_2^2 - (\sigma/3)\|\tilde{\Phi}\|_6^6] + \frac{\dot{a}^2}{4}I + \frac{\dot{a}}{a}\mathrm{Im}\int(\xi\tilde{\Phi}\partial_\xi\tilde{\Phi}^*)\,\mathrm{d}\xi. \tag{4.17}$$

Multiplying equation (4.12) by $\xi^2\tilde{\Phi}^*$ and integrating the imaginary part of the result over space leads to the vanishing of the last contribution of (4.17) with $\mathrm{Im}\int(\xi\tilde{\Phi}\partial_\xi\tilde{\Phi}^*)\,\mathrm{d}\xi = 0$. Expanding finally the virial relation $\partial_t^2(a^2 I) = 8H$ therefore restores the dynamical equation (4.15). It can be noticed in this respect that imposing the conservation of H thereby implies $(1/a^2)E\{\tilde{\Phi}\} \leqslant 0$ with $(\dot{a}^2/4) > 0$.

In order to know if the derivative term χ and the Raman term Γ in the virial identity (2.12) may affect the evolution of $a(t)$ as compared with the CNLS case, we simply repeat both of the former procedures, concentrating our attention on the case $\Gamma = \gamma = 0$ first (without the Raman contribution). In this case, equation (2.1) is a Lagrangian system with the density

$$\mathcal{L}_{\mathrm{EDNLS}} = \mathcal{L}_{\mathrm{CNLS}} + \tfrac{1}{4}\mathrm{i}|u|^2(u\partial_x u^* - u^*\partial_x u) = -\tfrac{1}{2}\mathrm{i}(u^*\partial_t u - u\partial_t u^*) - U_3 \tag{4.18}$$

with $H = \int U_3\,\mathrm{d}x$. Here, the additional contribution in $\mathrm{Im}(|u|^2 u\partial_x u^*)$ of (4.18) implies a modification of $\overline{L}(a, \dot{a}, \ddot{a})$ computed from $L_{\mathrm{EDNLS}} = \int \mathcal{L}_{\mathrm{EDNLS}}\,\mathrm{d}x$ as follows,

$$\overline{L} = -\frac{a\ddot{a}}{4}I - \frac{E_D\{\tilde{\Phi}\}}{a^2} - \frac{\dot{a}}{4a}\int \xi|\tilde{\Phi}|^4\,\mathrm{d}\xi \tag{4.19}$$

in which $E_D\{\tilde{\Phi}\}$ reads as the integral H given by (2.7) and expressed in terms of $\tilde{\Phi}$, i.e.

$$E_D\{\tilde{\Phi}\} = \|\partial_\xi\tilde{\Phi}\|_2^2 + \tfrac{1}{2}\mathrm{Im}\int |\tilde{\Phi}|^2(\tilde{\Phi}\partial_\xi\tilde{\Phi}^*)\,\mathrm{d}\xi - (\sigma/3)\|\tilde{\Phi}\|_6^6. \tag{4.20}$$

Deriving the variational equations for $a(t)$ from (4.19), we easily deduce

$$\epsilon = -\frac{a^3\ddot{a}}{4} = -\frac{E_D\{\tilde{\Phi}\}}{I} \tag{4.21}$$

showing that the extra contribution—arising from the derivative term in equation (2.1)—does not play any role in the dynamics of $a(t)$. This leads to similar behaviours as in the 'free' CNLS case, namely $a(t) \sim \sqrt{t_c - t}$ whenever $E_D\{\tilde{\Phi}\} < 0$, or $a(t) \sim (t_c - t)$ for $E_D\{\tilde{\Phi}\} = 0$. In this latter case, the linear-in-time scaling law $a(t)$ satisfying $\ddot{a}(t) = 0$ may only suit for describing the asymptotic stage $a(t) \to 0$ of the collapsing evolution and when using the substitution (4.16), $\tilde{\Phi}$ can be seen to reach in this limit the zero-energy ground-state solution Φ studied in section 3.

On the other hand, when employing (4.10), equation (2.1) self-similarly transforms into

$$\epsilon\xi^2\tilde{\Phi} + \partial_\xi^2\tilde{\Phi} - \mathrm{i}|\tilde{\Phi}|^2\partial_\xi\tilde{\Phi} - \tfrac{1}{2}a\dot{a}\xi|\tilde{\Phi}|^2\tilde{\Phi} + \sigma|\tilde{\Phi}|^4\tilde{\Phi} = 0 \tag{4.22}$$

from which the relation $\int \mathrm{Im}(\xi\tilde{\Phi}\partial_\xi\tilde{\Phi}^*)\,\mathrm{d}\xi = -\frac{1}{4}\int\xi|\tilde{\Phi}|^4\,\mathrm{d}\xi$ follows. Using this, the Hamiltonian (2.7) simplifies in turn as

$$H \to \frac{1}{a^2}E_D\{\tilde{\Phi}\} + \frac{\dot{a}^2}{4}I \tag{4.23}$$

so that the virial relation (2.12) finally restores the dynamical system (4.21). Estimated from the self-similar ansatz (4.10), the effect introduced by the nonlinear derivative term in equation (2.1) therefore does not affect the collapse dynamics already characterizing solutions to the CNLS equation.

Now regarding the Raman effect ($\gamma \neq 0$), we cannot use the Lagrangian approach in a convenient way, as equation (1.7) does not constitute a Hamiltonian system. However, we can again invoke the virial identity (2.12) together with the properties of $\tilde{\Phi}$ governed by equation (4.22) whose left-hand side now contains the extra contribution $\gamma\tilde{\Phi}\partial_\xi(|\tilde{\Phi}|^2)$ to read on the whole

$$\epsilon\xi^2\tilde{\Phi} + \partial_\xi^2\tilde{\Phi} - \mathrm{i}|\tilde{\Phi}|^2\partial_\xi\tilde{\Phi} - \tfrac{1}{2}a\dot{a}\xi|\tilde{\Phi}|^2\tilde{\Phi} + \sigma|\tilde{\Phi}|^4\tilde{\Phi} + \gamma\tilde{\Phi}\partial_\xi(|\tilde{\Phi}|^2) = 0. \tag{4.24}$$

When $\gamma \neq 0$, the virial (2.12) reading $\partial_t^2(a^2 I) = 8H$ for $\gamma = 0$ must consistently be supplemented by an additional contribution of the form $\tilde{\Gamma}/a^2$ to yield eventually

$$2a\ddot{a}I = (8E_D\{\tilde{\Phi}\} + \tilde{\Gamma})/a^2 \tag{4.25}$$

with $\tilde{\Gamma} = -4\gamma \int \xi\{\partial_\xi(|\tilde{\Phi}|^2)\}^2\,\mathrm{d}\xi$.

A spreading of the solution $u(x,t)$ can thus be inferred from the dynamical system (4.25) with $a(t) \sim t$ if $\tilde{\Gamma} = -8E_D\{\tilde{\Phi}\}$ or with $a(t) \sim \sqrt{t}$ if the Raman effect can be sufficiently strong to ensure $\tilde{\Gamma} > -8E_D\{\tilde{\Phi}\}$, which should hereby prevent a self-similar collapse. Note, however, that due to the self-similarity assumption according to which $\tilde{\Phi}$ has to remain time-independent, the linear-in-time scaling $a(t) \sim t$ leading to $a\dot{a} \sim t$ in equation (4.24) should in principle be forbidden.

This arrest of collapse induced by a spreading dynamics appears in a certain sense to be consistent with the property following which there is no localized ground-state solution when $\gamma \neq 0$. Indeed, keeping in mind that the collapse consists in a localizing process of nonlinear wavepackets whose most elementary shape $\tilde{\Phi}$ is sought under the form of zero-energy ground states, the non-existence of such localized states considerably limits the possibility of blow-up together with the possibility of realizing localized collapsing solutions to equation (1.7).

5. Validity of the variational approach

All the above arguments are based on an exactly self-similar behaviour of the solution $u(x,t)$ assumed to be self-similar and localized even in the presence of the derivative and Raman contributions. They only give some global tendencies of the time evolution of the singular part of the solution, which should be numerically checked in future investigations. In this respect, it is worthwhile noticing that in the presence of the derivative/Raman terms, the choice of the test function (4.10) could appear to be invalid regarding the space dependence of the phase: indeed, the latter functional phase dependence of the form $\exp[-\mathrm{i}(a\dot{a}/4)\xi^2]$ in (4.10) is always true for the simple case of the CNLS equation, as it can be verified by repeating the Lagrangian procedure with a more general test function

$$u(x,t) = \frac{1}{\sqrt{a(t)}}\tilde{\Phi}\left(\frac{x}{a(t)}\right)\exp\left[\mathrm{i}\frac{\theta(t)}{4}\frac{x^2}{a^2(t)}\right] \tag{5.1}$$

containing, for example, a real profile trial function $\tilde{\Phi}$. In the present context of the EDNLS equation, where $\tilde{\Phi}$ is not *a priori* determined (it can be complex valued), one can check from the variational method that $\overline{L}$ should be extended to a general function $L = \overline{L}(a, \dot{a}, \theta, \dot{\theta})$ yielding, through the variational equations $\delta\overline{L}/\delta\theta = \delta\overline{L}/\delta a = 0$,

$$\theta = -a\dot{a} - \frac{2}{I}(\mathcal{J} - \mathcal{B}/4) \tag{5.2}$$

with $\mathcal{J} = \mathrm{Im}\int(\xi\tilde{\Phi}^*\partial_\xi\tilde{\Phi})\,\mathrm{d}\xi$ and $\mathcal{B} = \int \xi|\tilde{\Phi}|^4\,\mathrm{d}\xi$, instead of $\theta = -a\dot{a}$ simply, and

$$\epsilon = -\frac{a^3\ddot{a}}{4} = -\frac{E_D\{\tilde{\Phi}\}}{I} + \frac{(\mathcal{J} - \mathcal{B}/4)^2}{I^2}. \tag{5.3}$$

From equation (5.3), it could be concluded that a more general conjugate momentum $\theta(t)$ including a constant contribution affects the behaviour of $a(t)$ by adding a negative quantity in the dynamical equation governing $\ddot{a}$. Apparently, this extra contribution, scaling as $E_D\{\tilde{\Phi}\}$, could be thought to strengthen the collapse. In reality, we have to account for the fact that changing $\theta(t) = -\dot{a}a$ into (5.2) also affects the self-similar solution $\tilde{\Phi}$, whose phase then exhibits an additional contribution quadratic in space with $\tilde{\Phi} \to \tilde{\Phi} \exp \mathrm{i}[(\mathcal{J} - \mathcal{B}/4)\xi^2/(2I)]$, and this necessarily changes the value of the energy integral $E_D\{\tilde{\Phi}\}$, so that, formally speaking, the dynamical system (5.3) has not to differ from (4.21) on the whole. Therefore, imposing $\theta(t)$ in the basic form $\theta(t) = -\dot{a}a$ instead of its more general counterpart (5.2) does not alter the previous results. This choice is, moreover, consistent with the mass continuity relation (2.2) of the full R-EDNLS equation (1.7), in the sense that it preserves this conservation law when the latter is expressed in terms of the function $\tilde{\Phi}$ through the substitution (5.1).

Furthermore, as seen from expression (2.11), the centre of mass of solutions to the R-EDNLS equation, whatever the value of γ may be, does not satisfy $N\partial_t^2\langle x\rangle = 0$, unlike the standard solutions to the CNLS equation (4.1). This means that the solutions to equation (1.7) may displace along the x-axis while they evolve towards a collapse or spread out. Such a displacement already characterizes the travelling-wave solutions moving with a constant velocity, identified in [7]. Taking this property into account, we can wonder whether this motion of nonlinear structures may affect their global dynamical behaviour previously estimated from steady-state trial functions. To answer this question partly, a more detailed analysis of the mean square radius $\langle x^2\rangle$ would have consisted of working on the complete virial identity—also called 'moment of inertia'—$\mathcal{M} \equiv N\langle(x - \langle x\rangle)^2\rangle$, which includes the motion of the CM of localized waveforms (see, e.g., [10]). Studying this quantity with the test functions (4.10) or (5.1), we can easily see that the integral $\mathcal{M} = N(\langle x^2\rangle - \langle x\rangle^2)$, carrying the additional contribution $\langle x\rangle^2 \sim a^2(t)$, may only modify at the most the value, but not the sign, of the coefficient on the right-hand side of equation (4.21) for $\gamma = 0$, or (4.25) in the opposite case $\gamma \neq 0$. Thereby, a first conclusion inferred from this straightforward argument is that the global evolution of waveforms governed by the R-EDNLS equation should not be altered significantly by the motion of the centre of mass. However, this conclusion is true provided that the trial solution (4.10) remains reliable when investigating the dynamics of the CM. In fact, when looking at solutions whose maximum displaces along the x-axis, we need to consider test functions characterized by an amplitude of the form $|u(x,t)| = [a(t)]^{-\frac{1}{2}}\,|\tilde{\Phi}(\tilde{\xi}, t)|$, where $\tilde{\xi} \equiv (x - x_0(t))/a(t)$ now contains the time-dependent coordinate $x_0(t)$ at which the waveform is expected to reach a maximum. Analysing such moving structures exceeds the limits of the present paper. Nevertheless, simple physical arguments allow us to guess that in the case $\gamma = 0$, the collapse dynamics may not be affected by the motion of the CM provided that near the singularity t_c the speed $\dot{x}_0$ is much smaller than the collapse velocity $\dot{a}$, which is quite possible for collapse rates characterized by the scaling law $a(t) \sim \sqrt{t_c - t}$. More precisely, repeating the Lagrangian procedure on these moving trial solutions shows that the dynamics of the CM becomes negligible in the system (4.21) if near the collapse singularity the time-dependent quantities $x_0(t)$ and $a(t)$ satisfy the inequality $|\ddot{x}_0|, |\dot{x}_0|/a^2 \ll |\ddot{a}|$. Under these conditions, the shape of a waveform travelling along the x-axis may undergo a nonlinear steepening due to the nonlinear derivative in the EDNLS equation, but this steepening should not prevent the waveform from collapsing ultimately at a finite time t_c. In the vicinity of this instant, the spiky core of a collapsing waveform can therefore be modelled to a first approximation by means of the self-similar solution (4.10).

Besides, when collapse occurs (i.e. for $\gamma = 0$) with a scaling law $a(t) \sim \sqrt{t_c - t}$ implying $\epsilon = \text{constant} > 0$, the self-similarity assumption (4.10) is true up to some logarithmic spatial divergences of the L^2 norm, which here have been disregarded: indeed, as the nonlinear contributions in $|\tilde{\Phi}|^2$ are localizing potentials that algebraically decay as $\xi \to \infty$, we can infer from equation (4.22)—as well as from the transformed CNLS equation (4.12)—that for $\epsilon > 0$, $\tilde{\Phi}$ should asymptotically behave as $\tilde{\Phi} \sim 1/\sqrt{\xi}$ at large distances $\xi \rightarrow +\infty$, leading to a logarithmic divergence in N. This divergence can be removed in the context of CNLS by considering so-called quasi-self-similar solutions $\tilde{\Phi}(x/a(t), \epsilon(t))$ that contain by themselves an adiabatic dependence on time through the function $\epsilon(t)$ assumed to vanish very slowly in time. To solve this problem, it is then necessary to keep the time derivative in the transformed CNLS by making use of the general substitution

$$u(x,t) = \frac{1}{\sqrt{a}}\tilde{\Phi}(\xi,\tau)\exp\left[-\mathrm{i}\lambda\tau - \mathrm{i}\tfrac{1}{4}a\dot{a}\xi^2\right]$$

to get

$$-\mathrm{i}\partial_\tau\tilde{\Phi} + (\epsilon\xi^2 - \lambda)\tilde{\Phi} + \partial_\xi^2\tilde{\Phi} + \sigma|\tilde{\Phi}|^4\tilde{\Phi} = 0 \tag{5.4}$$

with $\tau(t) \equiv \int_0^t \mathrm{d}u/a^2(u)$. Multiplying (5.4) by $\tilde{\Phi}^*$ and retaining the imaginary part of the result, we get the mass continuity equation

$$\int_0^{\xi\to+\infty} \partial_\tau(|\tilde{\Phi}|^2)\,\mathrm{d}\xi = 2\mathrm{Im}(\tilde{\Phi}\partial_\xi\tilde{\Phi}^*)|_{\xi\to+\infty} \tag{5.5}$$

whose right-hand side is estimated from (5.4) under the quasi-self-similar assumption $\partial_\tau\tilde{\Phi} \rightarrow 0$. At large distance $\xi \rightarrow \infty$, the localizing nonlinearity $|\tilde{\Phi}|^4$ vanishes and a simple BKW analysis of (5.4) leads to

$$\tilde{\Phi} \underset{\xi\to\infty}{\sim} \exp[-\pi\lambda/4\sqrt{\epsilon}]\exp\left[-\mathrm{i}\frac{\sqrt{\epsilon}}{2}\xi^2\right]/\sqrt{\xi}.$$

Inserting this asymptotics into (5.5) and making use of the Taylor expansion $\tilde{\Phi} = \tilde{\Phi}(\epsilon = 0) + \epsilon\tilde{\Phi}_\epsilon(\epsilon = 0) + \cdots$ yields

$$\epsilon_\tau = -2\exp[-\pi\lambda/2\sqrt{\epsilon}]$$

from which the 'adiabatic' behaviour

$$\epsilon(t) = \pi^2\lambda^2/4\left(\ln\left(\ln\left(\frac{1}{t_c - t}\right)\right)\right)^2 \tag{5.6}$$

follows with $\tau \sim \ln(1/(t_c - t))$. The asymptotic time dependence (5.6) of $\epsilon(t)$ amounts to introducing a twice logarithmic correction in $a(t)$ (see for instance [12]). Repeating this quasi-self-similar analysis in the present context including the nonlinear derivative term exceeds the limit of this paper. However, regarding the self-similarly transformed problem (4.22), where the time derivative $-\mathrm{i}\partial_\tau\tilde{\Phi}$ must be restored, we can first check that the EDNLS equation (4.22) admits a mass continuity equation analogous to (5.5). Then, estimating the behaviour of $\tilde{\Phi}$ as $\xi \rightarrow \infty$ leads in principle to the same function as in the CNLS case under the constraint of a decaying nonlinearity $|\tilde{\Phi}|^2 \rightarrow 0$ as $\xi \rightarrow \infty$. Finally, equation (4.22) suggests the following Taylor expansion $\tilde{\Phi} = \tilde{\Phi}(b = 0) + b\tilde{\Phi}_b(b = 0) + \frac{1}{2}b^2\tilde{\Phi}_{bb}(b = 0) + \cdots$, with $b = -a\dot{a}$ (satisfying $b_\tau \rightarrow 0$ as $\tau \rightarrow +\infty$), which should yield $b \sim \pi\lambda/\ln\tau$ using (5.5) in the self-similar limit $\epsilon = (b^2 + b_\tau)/4 \rightarrow b^2/4$, and therefore lead to

$$a(t) \sim \left(\frac{2\pi\lambda(t_c - t)}{\ln(\ln(1/(t_c - t)))}\right)^{1/2} \tag{5.7}$$

as in the CNLS case.

6. Conclusion

To summarize, we have shown some general properties of stationary and self-similar solutions to the R-EDNLS equation (1.7). Among them, it has been proved that the Hamiltonian attached to the EDNLS equation and computed on the stationary ground states reduces to zero. However, unlike the solutions to the CNLS equation for which the virial identity (4.8) provides an exact proof of finite-time blow-up under the sufficient condition $H < 0$ implying $N > N_0$, solutions to the EDNLS equation have not been rigorously shown to collapse, due to the complexity of the virial identity (2.12). Thus, the negativity of the Hamiltonian alone cannot guarantee the blow-up of solutions to the EDNLS equation possessing a mass above the critical threshold N_c defined by (4.7), and the present results therefore require to be confirmed numerically. Using a Lagrangian approach based on a self-similar and localized trial solutions, we have nevertheless displayed that the derivative term in the EDNLS equation (2.1) may not stop the collapse. Even if it contributes to a dispersive effect, this nonlinear derivative term keeps the system Hamiltonian and conservative with a mass integral remaining preserved and involved in the collapse process. Adding the Raman response, we have shown that the resulting system does not admit any localized ground-state solutions and that the Raman effect can compete with and eventually arrest the collapse.

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